THE PROBLEM OF DISCRETIZATION FOR DYNAMIC ANALYSIS

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Keywords: dynamic analysis, discretization, lattice-vibration, numerical dispersion

Introduction

Today, numerical analysis and a calculation based on digital data or the numerical model are essential technology for structural engineering. A continuous signal or the continuum system includes an infinite number of values in any short time or any small space. Digitalization of them is reduction or thinning-out into a limited number of values. This process is called sampling or discretization and causes an information loss of original signal or system. However, according to Nyquist sampling theorem, perfect reconstruction is possible from discrete signals which are sampled at a rate more than twice of the highest frequency of the original continuous signal. Then, before sampling a continuous signal must be cut off any frequency higher than 1/2 of the sampling rate (that is called Nyquist rate) by a low-pass filter or an anti-aliasing filter. Therefore, a reconstructed signal can be represented approximately by Fourier polynomial (or a convolution of discrete data with a periodic sinc function) not including higher components than Nyquist frequency, and it is mathematically continuous and smooth.

Time-discretization

Ground motion of earthquake was digitally recorded usually at a sampling rate of 100Hz in accordance with the sampling theorem. Digital reconstruction of intermediate data in the sampling interval is called up-sampling. Fig.1 shows a part of reconstructed data from the UD acceleration recorded at the IWTH25 (Ichinoseki-nishi) station of KiK-net (Strong motion sensor networks in Japan) on Jun. 14, 2008 which is known as the Guinness World record. The up-sampled data are obtained by interpolation using the FFT (First Fourier Transform).

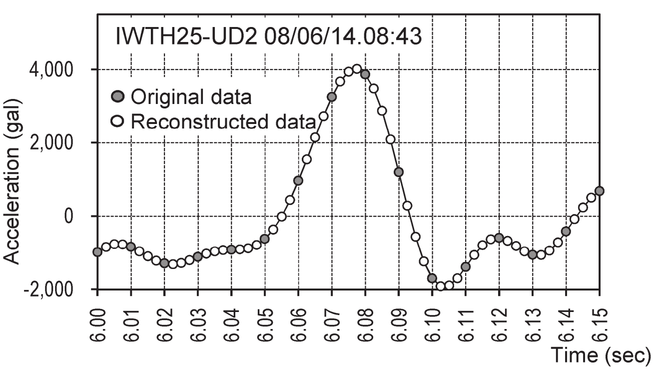


Fig. 1: A part of reconstructed data from the UD acceleration records at IWTH25-station on 2008/06/14 (13sec time-shifted)

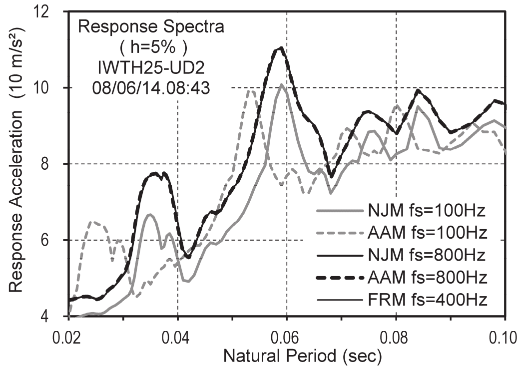


Fig. 2: Response Spectra of IWTH25-UD2, 08/06/14, 08:43 caluculated by 3 metods

The peak ground acceleration (PGA) of 4,015.8 gal can be find at intermediary of sampling data and it is lager than “PGA = 3,866 gal” in the document [3]. Fig. 2 shows response spectra of this sismic wave which are calculated by 3 methods of average-acceleration method (AAM), Nigam-Jennings method (NJM) and frequency-response-function method (FRM). And sampling frequencies (f_s) of input data are 100Hz of the original data, 400Hz or 800Hz of up-sampled data by FFT interpolation. In the case of f_s = 800Hz, response spectra calculated respectively by 3 methods are approxmately equal. Those response at T=0.06sec are nearly 9% lager than “9,853 cm/s² at T = 0.06sec” in the document [4].

Therefor, up-sampling of an input seismic wave is recommended for dynamic analysis of a structure with high natural frequency. On the other hand, it seems as if reconstruction or interporation based on the sampling theorem would be inconsistent with “the law of casuality”, because the periodic function (which is reconstucted) repeats its values in a certain period.

Space-discretization

The above-mentioned studies are about problems of time-discretization. There are also problems of space-discretization or discrete models of continuum system. The most simplified model of the continuum system is a homogeneous linear-elastic bar with fixed ends. It is a natural modeling of continuum system to decompose into small elements of the inertial mass or elasticity. Lattice-vibration model shown in Table 1 is primitive and classical but it is same as the FEM (Finite Element Method) model of one-dimensional (1D) system.

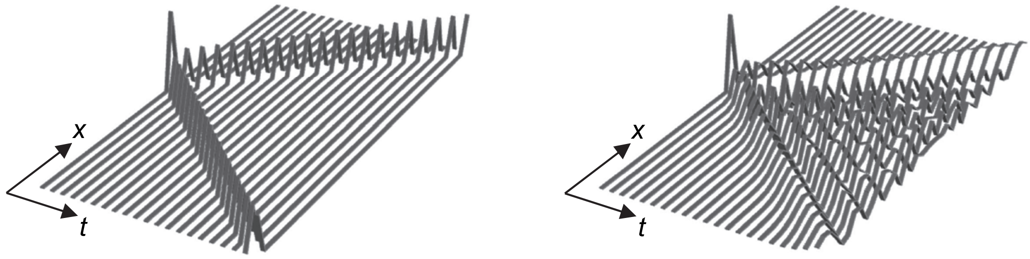
Table 1: 1D continuum model and lattice model

	Continuum Model	Lattice Model
Model	 Length : L, density : ρ Stiffness : μ	 Spacing : $d=L/N$ Mass : $m=\rho\, d$, Spring : $k=\mu/d$
Motion Eq.	$\dfrac{\partial^2 u}{\partial t^2}=c^2\dfrac{\partial^2 u}{\partial x^2}$ $c^2=\mu/\rho=\mathrm{const.}$	$\ddot{u}_j=c^2\dfrac{u_{j-1}-2u_j+u_{j+1}}{d^2}$ $\rightarrow\ m\boldsymbol{u}=-\boldsymbol{K}\boldsymbol{u}$
Normal Modes	$q_n(x)=\sin(\kappa_n x)$ $\kappa_n=n\pi/L$	$\boldsymbol{q_n}=\sqrt{2/N}\,[\sin(\kappa_n j\, d)]$ $1\leq n,\ j\leq N-1$
Natural Frequency	$\omega_n=\kappa_n c=n\pi\dfrac{c}{L}$	$\omega'_n=\dfrac{2Nc}{L}\sin(\dfrac{n\pi}{2N})$

The dynamic properties of the system are characterized by natural frequencies and normal modes depending on material properties of L, ρ and μ. All of normal modes of the lattice model match those of the continuum model not more than the N-th mode of vibration. However, the n-th natural frequency ω'_n of the lattice model is less than ω_n of the continuum model. The higher the frequency is, the greater is the difference of them. This phenomenon is called “dispersion relation” or “numerical dispersion”. The free vibration is expressed by Eq. (1) when the initial condition is a periodic sinc function.

$$u(t,x)=\frac{2}{N}\sum_{n=1}^{N/2}\sin(\kappa_{2n-1}x)\cos(\omega_{2n-1}t),\quad u(0,x)=\frac{2}{N}\sum_{n=1}^{N/2}\sin(\kappa_{2n-1}x) \qquad \text{Eq. (1)}$$

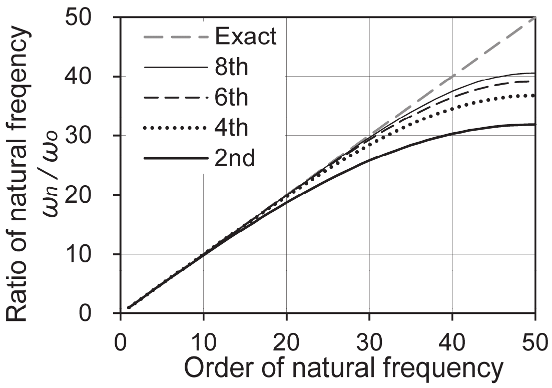
In addition, it is the wave propagation as shown in Fig. 3.



The continuum model without dispersion The Lattice model with dispersion

Fig. 3: Difference of wave propagation between 2 models

On the other hand, the lattice model is also same as FDM (Finite Difference Method) model of 1D system. There are a number of finite-difference approximations of high-order accuracy shown as below. Even if they are applied, the normal modes are same as the lattice model (because of the symmetry of the coefficients). However, the higher the order of accuracy is, the dispersion is small, as shown in Fig. 4.



The 2nd-order derivative approxmation by central finite difference of higher order accuracy

$u''_i\, d^2 \approx \qquad u''(x)=\dfrac{\partial^2 u}{\partial x^2}$

2nd: $-2u_i + u_{i\pm 1}$

4th: $-\dfrac{5}{2}u_i + \dfrac{4}{3}u_{i\pm 1} - \dfrac{1}{12}u_{i\pm 2}$

6th: $-\dfrac{49}{18}u_i + \dfrac{3}{2}u_{i\pm 1} - \dfrac{3}{20}u_{i\pm 2} + \dfrac{1}{90}u_{i\pm 3}$

8th: $-\dfrac{205}{72}u_i + \dfrac{8}{5}u_{i\pm 1} - \dfrac{1}{5}u_{i\pm 2} + \dfrac{8}{315}u_{i\pm 3} + \dfrac{7}{240}u_{i\pm 4}$

Fig. 4: Dispersion relation of FDM (N=50) used approxmation of higher order accuracy

When FDM is represented by a matrix-vector equation, the bandwidth of stiffness matrix is larger as the order of accuracy is higher. In addition, the stiffness matrix reconstructed by the normal modes and the natural frequencies of the continuum model is a full matrix. The stiffness matrix of the lattice model is tri-diagonal matrix which shows “local-action theory”. Therefore, it shows nonlocal action that the bandwidth of the stiffness matrix is more than 3.

Fig. 5 shows a 1D model of multi-layered soil and its normal modes (or stationary waves) of the 1st to 4th order. This model is formed by jointing several homogenous bars. When it vibrates at a natural frequency ω_n, the mode shape of each bar is a part of sinusoidal curve same as that of a homogenous bar. The natural frequencies and normal modes can be determined to satisfy boundary conditions of each layer (in the same manner as multi-reflection theory). And an apparent wave number in each layer can be equalized by scaling the depth of layer. In the same manner as Fig.3, Fig. 6 shows a wave propagation in the medium formed by jointing symmetrically 2 models in Fig. 5. And, in this model, the length of each bar is scaled. Reflected waves on the boundary can be find.

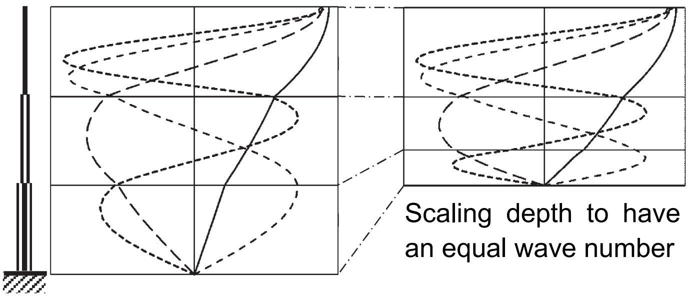


Fig. 5: Normal modes on a 1D model of multi-layered soil

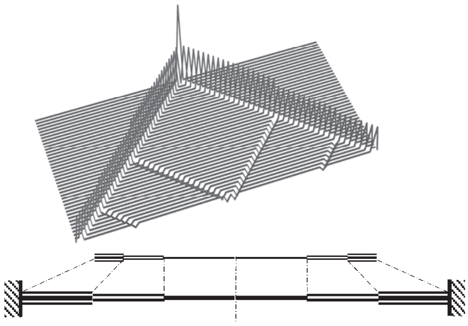


Fig. 6: Wave propagation in a multi-bar system (without dispersion)

In the lattice model of the above system, when the lattice spacing (or mesh size) d_i is proportional to the wave velocity c_i of the i-th layer (that is d_i / c_i = constant) and the boundaries are also at the lattice points, all normal modes of the lattice model match to those of the continuum model. However, their natural frequencies do not match to those of the continuum model and there is a “dispersion relation” same as the simply model.

Summery

1) Discretization causes the essential difference and that cannot be quantitatively compensated always.

2) In some case, up-sampling of the digital record of seismic wave is necessary before the time-history response analysis.

3) The high mode vibration cause numerical dispersion in the space-discrete model formed by connecting adjacent nodal points by elastic elements. Such a discretization is also based on the local-action.

4) This study shows the problem of a simple model of linear one-dimensional. However, in a complex model as non-linear or three-dimensional, it might be other problems exist.

References

[1] H. Kogo & T. Mita, Introduction to System Control, Jikkyo Publishing Co., 1979. (In Japanese)

[2] M. Toda, Theory of Vibration, Baifukan Publishing Co., 1968. (In Japanese)

[3] http://www.kyoshin.bosai.go.jp/kyoshin/topics/Iwatemiyaginairiku_080614/IWTH25_NIED.pdf (In Japanese)

[4] http://www.kyoshin.bosai.go.jp/kyoshin/topics/Iwatemiyaginairiku_080614/IWT_spectra.pdf (In Japanese)

[5] S. Tosu, The Problem of Time Discretization for Dynamic Analysis, Proceedings of research meetings of Architectural Institute of Japan, Kinki Branch, 2015. (In Japanese)

[6] Wikipedia (English site)